Basic Facts

2022年1月25日 23:10

Without mentioning, G ℓ-group, all rep being smooth

* Schur's LM

Hypothesis: Countable at infinity: For $\forall$ cpt open subgp $K \subseteq G$, G/K is countable

Rmk: If this holds for one cpt open subgp K, then it holds for all cpt open subgp K'. Because $K \cap K'$ is cpt open, of finite index in K

$G/K\cap K' \twoheadrightarrow G/K$ $\quad G/K\cap K' \twoheadrightarrow G/K' \Rightarrow G/K'$ countable

LM1: (π, V) an irr rep ~~rep~~ of $G \Rightarrow \dim_{\mathbb{C}} V$ should be countable

Pf: take $0 \neq v \in V$, it's fixed by K (cpt) open subgp.

$\{\pi(g)v : g \in G/K\}$ will generate V. $\square$

Schur's LM: (π, V) irr smooth rep $\Rightarrow \mathrm{End}_G(V) = \mathbb{C}$

Pf: $\underline{0 \neq \phi} \in \mathrm{End}_G(V) \Rightarrow \mathrm{im}\,\phi = V$, $\ker \phi = 0$

$\Rightarrow \phi$ is invertible $\Rightarrow \mathrm{End}_G V$ is complex division algebra.

Then $0 \neq v \in V$, we know that G-translations of v will span V

$\Rightarrow$ ~~also~~ ϕ will determined by $\phi(v)$.

By LM1, $\Rightarrow \mathrm{End}_G(V)$ has countable dimension over $\mathbb{C}$

Take $\phi \in \mathrm{End}_G(V) - \mathbb{C}$, it's transcendental over $\mathbb{C}$

we can gerate $\mathbb{C} \subsetneq \mathbb{C}(\phi) \subseteq \mathrm{End}_G(V)$

But then $\{(\phi - a)^{-1} : a \in \mathbb{C}\} \subseteq \mathbb{C}(\phi)$ linearly independent over $\mathbb{C}$

uncountable $\square$

Rmk: Alg clos of $\mathbb{C}$. To any field which is (uncountable) alg closed.

Rmk: When G is cpt (profinite group), the converse of the Schur's LM also holds. $V \cong \bigoplus V_i$

$\mathrm{End}_G(V)$ is 1-dimensional iff V is irr

Rmk: The above does not hold in general if G is not cpt

Eg: $G = GL_2(F)$ wher F is a local field (non arch)

1_T $\quad T$ is ~~a~~ maximal torus

$\mathrm{End}_G \mathrm{Ind}_T^G 1_T$ is 1-dimensional $\quad \mathrm{Ind}_T^G 1_T$ is not irr

Def: A character of a ℓ-gp G is a continuous homomorphism from $G \longrightarrow \mathbb{C}^\times$

Eg: 1_G.

Prop (Criterion): Let $\psi: G \to \mathbb{C}^\times$ be a group homo
then the following two conditions are equiv.
① ψ is continuous
② $\ker\psi$ ~~sta~~ is open

Pf: ② $\Rightarrow$ ①
① $\Rightarrow$ ② N open nbd of 1 in $\mathbb{C}^\times$
N small enough to ensure no subgroup of $\mathbb{C}^\times$ is contained in N
then $\psi^{-1}(N)$ is open $\Rightarrow K \subseteq \ker\psi$ it's open $\square$

Def: ψ be a character. If $\operatorname{im}\psi \subseteq \{|z|=1\} \subseteq \mathbb{C}^\times$
call it unitary

Prop: If G is countable at infinite & ψ is a charater
$\Rightarrow \psi$ is a unitary charater.

Pf: S^1 is the unique maximal cpt subgp of $\mathbb{C}^\times$
K cpt $\Rightarrow \psi(K)$ cpt $\subseteq S^1$ $\square$

Cor of Shur's LM: (π, V) irr smooth rep of G, Let Z be the center of G
then Z acts on V by a character of Z,
$\pi(z)\cdot v = \omega_\pi(z)\cdot v$ where $z \in Z$, $v \in V$
$\omega_\pi(\ )$ is the chacrater of Z.

Pf: By Schur's LM, we have a homo from $Z \to \mathbb{C}^\times$
$\omega_\pi(z)$
Continuous: If we take cpt open subgp K such that
$V^K \neq 0 \Rightarrow \omega_\pi|_{K\cap Z} \equiv 1$
$\Rightarrow$ it's continuous $\square$

Cor: If G is abelian, then any irr smooth rep (π, V)
$\Rightarrow \pi$ is 1-dimensional.

Duality (Additive) F local field, non-arch $\widehat{F}$ be the group of charaters under the multiplication. $\mathfrak{p}$ prime ideal of $\mathbb{F}$

then $\{\mathfrak{p}^n\}_{n\in\mathbb{Z}}$ character

Def: the level of ψ is the least integer n

such that $\mathfrak{p}^d \subseteq \ker\psi$

$\Leftrightarrow \psi|_{\mathfrak{p}^d}$ is trivial

Prop: Let $\psi \in \hat{\mathbb{F}}$, $\psi \neq 1$, level of ψ is d

① $a \in \mathbb{F}$, $a\psi : \mathbb{F} \to \mathbb{C}$ $\quad x \mapsto \psi(ax)$

it's a character of $\mathbb{F}$.

And if $a \neq 0 \Rightarrow a\psi$ has level $d - v_{\mathbb{F}}(a)$. □

② $\mathbb{F} \to \hat{\mathbb{F}}$ $\quad a \longmapsto a\psi$ is a group isomorphism

Pf: injectivity is immediate

surjectivity: $\theta \in \hat{\mathbb{F}}$, $\theta \neq 1$ with level.

Let π be a prime element of $\mathbb{F}$, $u \in U_F$

then $u\pi^{d-\ell}\psi$ should have level ℓ

And then it agree on $\mathfrak{p}^\ell$

$\mathfrak{p}^{\ell-1}$ $\quad \mathfrak{p}^{\ell-2}$ $\quad$ 也

$u\pi^{d-\ell}\psi$, $u'\pi^{d-\ell}\psi$ they will agree on $\mathfrak{p}^{\ell-1}$

iff $u \equiv u' \pmod{\mathfrak{p}}$ $\qquad q = \# \mathcal{O}/\mathfrak{p}$

there are only $q-1$ nontrivial character of $\mathfrak{p}^{\ell-1}$

which are trivial on $\mathfrak{p}^\ell$

Let u range over $U_F / U_F^1 \Rightarrow$ these character of $\mathfrak{p}^{\ell-1}$

should be distinct

So there $\exists\, u_1\pi^{d-\ell}\psi|_{\mathfrak{p}^{\ell-1}} = \theta|_{\mathfrak{p}^{\ell-1}}$

Proceed, $u_1, u_2, u_3, \ldots\ldots$

$u_i \equiv u_{i+1} \pmod{\mathfrak{p}^i} \Rightarrow \{u_i\} \to u$

such that $u\pi^{d-\ell}\psi$ agree with θ □

* Semi-simple G - ℓ-group.

Prop & Def: (π, V) smooth rep G. TFAE

① V is the sum of its irr G- subspaces

② V is the direct sum of a family of irr G-subspaces

③ For any G-subspace W of V, there $\exists$ a G-complement $W^\perp$ such that $W \oplus W^\perp \simeq V$

Pf: (Zorn's Lm) $\square$

is called semi-simple rep.

Prop: G ℓ-gp, K cpt open subgp. (π, V) smooth rep of G

$\Rightarrow$ V should K-semisimple. $V \xrightarrow{\sim} \bigoplus$ sum of irr K-subspaces of V.

Eg: G cpt, i.e. profinite. (π, V) is irr, K cpt open subgp

$\Rightarrow$ V is finite dimensional $\{\pi(g) v : g \in G/K\}$ is finite.

$K' = \bigcap_{g \in G/K} gKg^{-1} \Rightarrow K'$ is open, normal subgp of G (of finite index)

K' acts trivially on V

$\Rightarrow$ V can be view as an irr rep of the finite discrete G/K'

Pf of Prop: $v \in V \Rightarrow$ open cpt normal K' to fix v.

then it generate a finite-dimensional K-subspace W, K' acts on W trivially

W can be viewed as a rep of finite gp K/K'

$\Rightarrow$ it's sum of its irr K-subspaces

As v is arbitrary $\Rightarrow$ V is K-semisimple $\square$

Let $\hat{K}$ ~~be~~ the set of equiv classes of irr smooth rep of K.

If $\rho \in \hat{K}$, (π, V) smooth rep of G

$V^\rho :=$ sum of all irr K-subspaces of V in the class of ρ.

it's called the ρ-isotypic component of V.

Eg: V^K is the isotypic component of V in the class of 1.

Prop: With above notations (π, V) smooth rep of G (over K)

① V is the direct sum of its isotypic components

$V = \bigoplus_{\rho\in\hat{K}} V^\rho$

② Take (σ, W) smooth rep of G.

G-homo $f: V \to W$

then it map $f(V^\rho) \subseteq W^\rho$

& $W^\rho \cap f(V) = f(V^\rho)$

Pf: ① V is K-semisimple $\Rightarrow V = \bigoplus_{i\in I} U_i$ K-subspaces (irr)

now let $\underset{V^\rho}{\underset{\shortparallel}{\underline{U(\rho)}}}$ be the sum of those U_i of class ρ

$V = \bigoplus_{\rho\in\hat{K}} U(\rho)$. Take any W irr K-subspace

if it's of class $\rho \Rightarrow W \subseteq U(\rho)$

Else, there should be a non-zero K-homo from W to U_i of some class $\tau \neq \rho$

by irr $\Rightarrow W \subseteq U(\tau)$

$\Rightarrow V^\rho = U(\rho)$. ① $\square$.

image of V^ρ in W, it's a sum of irr K-subspace of W of class ρ $\Rightarrow \operatorname{im}(V^\rho) \subseteq W^\rho$

$f(V) =$ sum of images $f(V^\tau)$ $\tau\in\hat{K}$

$f(V^\tau) \subseteq W^\tau$

as the sum of W^τ should be direct

$\Rightarrow f(V) = \bigoplus_{\tau\in\hat{K}} f(V^\tau)$ $\square$

Cor: $\boxed{a: U \to V}$ / $b: V \to W$ $\quad U \xrightarrow{a} V \xrightarrow{b} W$

it's exact if and only if $\forall K$ cpt open subg

$U^K \xrightarrow{u} V^K \xrightarrow{v} W^K$ is exact.

$\square$

ρ 1 $V^\rho = V^K$

$H \subseteq G$ subgp, define $V(H) :=$ linear span of the set $\{v - \pi(h)v : h \in H, v \in H\}$

it's a H-subspace $\pi(h_1)(v - \pi(h_2)v)$

$= \boxed{\pi(h_1)v} - \pi(h_1 h_2 h_1^{-1})\boxed{\pi(h_1)v}$

Cor2: $V(K) = \boxed{\bigoplus_{\substack{\rho \in \hat{K} \\ \rho \neq 1}} V^\rho}$ $W :=$ $\Rightarrow$ $V(K) \oplus V^K = V.$

Pf: is the G-complement $V \twoheadrightarrow V^K$

the kernel is W. At first $V(K) \subseteq W$

$V(K)$ is containe in the kernel of any K-homo $V \to V^K$

On the other hand, if we take any irr K-subspace U of class $\rho \neq 1 \Rightarrow U(K) = U$

Take all U. $\Rightarrow V^\rho \subseteq V(K)$

$\Rightarrow W \subseteq V(K)$ $\square$

Induction & Compact Induction

G ℓ-gp. H closed subgp

(σ, W) smooth rep of H. $X : \{f : G \to W :$

$h \in H$, $g \in G$

① $f(hg) = \sigma(h) f(g)$

② (smooth) $\exists$ K cpt open subgp such that $f(gx) = f(g)$ $g \in G$, $x \in K$ $\}$

Then $\Sigma : G \to \mathrm{Aut}_{\mathbb{C}}(X)$

$g \longmapsto (x \longmapsto f(xg) \quad x \in G)$

(Σ, X) of G by $\mathrm{Ind}_H^G \sigma$

There is a canonical map: H-homo

$\alpha_\sigma : \mathrm{Ind}_H^G \sigma \longrightarrow W$

$f \longmapsto f(1)$

Prop (Frobeniu Reciprocity). (π, V) of G (σ, W) of H

The canonical map

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G \sigma) \to \mathrm{Hom}_H(\mathrm{Res}_H^G \pi, \sigma)$$

$$\phi \longmapsto \alpha_\sigma \circ \phi$$

Pf: To give the inverse map: namely, take $f: W \to V$ to be a H-homo

define $f_*: V \to X\left(\begin{array}{l} G \longrightarrow W \\ g \longmapsto f(\pi(g)v) \end{array} \right)$

$v \longmapsto$

□

Prop: $\mathrm{Ind}_H^G: \mathrm{Rep}(H) \to \mathrm{Rep}(G)$ is additive & exact.

Pf: (σ, W) of H. $I(\sigma) = \{ f: G \to W \mid$ such that ① holds$\}$

Then $I(\sigma)$ is an abstract rep space

it's both exact and additive

$\mathrm{Ind}_H^G \sigma = \boxed{I(\sigma)^\infty} = \bigcup_{K \text{ cpt open}} I(\sigma)^K$

it's left exact

Only need to show the right exactness:

Take H-surjection $f: W \to U$ (σ, W) H

(τ, U)

Take $\phi \in I(\tau)^\infty$, K cpt open fix ϕ

$\phi: X(\tau) \to U$ $\mathrm{supp}\,\phi = \bigcup HgK$

Take $g \ni \phi(g) \in U$ is fixed by $\tau(\underline{H \cap gKg^{-1}})$

H trivial rep of $H \cap gKg^{-1}$

$\exists\, w_g \in W$ such that it's fixed by $\sigma(H \cap gKg^{-1})$

$\boxed{f(w_g) = \phi(g)}$

$\Phi: G \to W$ $\mathrm{supp}\,\Phi = \mathrm{supp}\,\phi$

$\Phi(hgk) = \sigma(h) w_g$ $\forall g \in H \backslash \mathrm{supp}\,\phi / K$

is fixed by K, the image of Φ is ϕ

□

Def: ① $f: G \to W$ is compactly supported modulo H if there $\exists$ a cpt subset $C \subseteq G$ such that $\mathrm{supp}\, f \subseteq HC$

② (σ, W) H, $X \supseteq X_c$ consists of all functions $f \in X$ such that f is cpt supp modulo H

X_c is G-subspace, it's smooth

$c\text{-}\mathrm{Ind}_H^G \sigma : G \to \mathrm{Aut}_{\mathbb{C}}(X_c)$.

there $\exists$ a canonical embedding $c\text{-}\mathrm{Ind}_H^G \to \mathrm{Ind}_H^G$

Prop: $c\text{-}\mathrm{Ind}_H^G$ is additive and exact.

H open subgp. canonical map $\alpha_\sigma^c : W \to c\text{-}\mathrm{Ind}\,\sigma$
$w \mapsto f_w$
$\mathrm{supp} f_w \subseteq H$, $f_w(h) = \sigma(h).w$ $h \in H$

Thm (Frob-Reciprocity). (σ, W) H, (π, V) G.
The canonical map $\mathrm{Hom}_G(c\text{-}\mathrm{Ind}_H^G \sigma, \pi) \to \mathrm{Hom}_H(\sigma, \mathrm{Res}_H^G \pi)$
$f \longmapsto f \circ \alpha_\sigma^c$.

Pf: To define the inverse map namely, take $\Phi : W \to V$ as an H-homo
there $\exists$ unique G-homo $\Phi_* : X_c \to V$
$f_w \mapsto \Phi(w)$ for any $w \in W$

$\Phi \mapsto \Phi_*$ $\square$

Matrix Elements

Dual Rep: (π, V) of G, $V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$
$V^* \times V \to \mathbb{C}$ $(v^*, v) \mapsto \langle v^*, v \rangle \in \mathbb{C}$

So we define a rep (π^*, V^*) as follows:
$\langle \pi^*(g) v^*, v \rangle := \langle v^*, \pi(g^{-1}) v \rangle$

it's not in general smooth. We take the smooth part of this rep. $\check{V} := (V^*)^\infty$

$\check{\pi} = \pi^* |_{\check{V}}$

$(\check{\pi}, \check{V})$ a smooth rep of G

$\langle \check{\pi}(g)\check{v}, v \rangle = \langle \check{v}, \pi(g^{-1})v \rangle$

Fix K cpt open $G \Rightarrow V^K$ has unique complement $V(K)$

If $\check{v} \in \check{V}$ is fixed by K (namely, $\check{v} \in \check{V}^K$)

then $\langle \check{v}, V(K) \rangle = 0$

$\check{v}$ is determined by the action of $\check{v}$ on V^K

Prop: $\check{V}^K \xrightarrow{\sim} (V^K)^*$

Pf: Extend a function on V^K.

to be an element of $\check{V}^K$ by

letting the value to be zero on $V(K)$ $\square$

Cor: (π, V) smooth rep of G. $0 \neq v \in V$

Then there $\exists$ a $\check{v} \in \check{V}$ such that $\langle \check{v}, v \rangle \neq 0$

Pf: $\exists K$ fix $v \in V^K \Rightarrow f \in (V^K)^*$ $f(v) \neq 0$

$\Rightarrow f \in \check{V}^K$ $\square$

Prop: $\delta: V \to \check{\check{V}}$ is an iso iff π is admissible

$\langle \delta(v), \check{v} \rangle_{\check{V}} = \langle \check{v}, v \rangle_V$. it's injective

Pf: K $\delta^K: V^K \to \check{\check{V}}^K$

δ is surj $\Leftrightarrow$ δ^K is surj for $\forall K$

$\Leftrightarrow$ V^K is finite-dim

$\Leftrightarrow$ π is admissible

Matrix Elements

(π, V) of G, $v \in V$, $\check{v} \in \check{V}$.

function $\gamma_{\check{v} \otimes v} : G \longrightarrow \mathbb{C}$

$g \longmapsto \langle \check{v}, \pi(g) v \rangle$

Define $C(\pi)$ to be linear span of $\{\gamma_{\check{v} \otimes v} \mid \check{v} \in \check{V}, v \in V\}$

Then $f \in C(\pi)$ is called the matrix coefficient of π.

Rmk:

Irr smooth rep of $GL_n(F)$ is addimible

$GL_2(F)$.

① Induced

② $\boxed{\text{cuspidal}} \longleftrightarrow \boxed{\gamma\text{-cuspidal}}$ admissible

$V_N = V / V(N)$